

$\phi_\epsilon$ -COORDINATED MODULES FOR VERTEX ALGEBRAS

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ABSTRACT. We study  $\phi_\epsilon$ -coordinated modules for vertex algebras, where  $\phi_\epsilon$  with  $\epsilon$  an integer parameter is a family of associates of the one-dimensional additive formal group. As the main results, we obtain a Jacobi type identity and a commutator formula for  $\phi_\epsilon$ -coordinated modules. We then use these results to study  $\phi_\epsilon$ -coordinated modules for vertex algebras associated to Novikov algebras by Primc.

## 1. INTRODUCTION

In a program to associate quantum vertex algebras to quantum affine algebras, a theory of  $\phi$ -coordinated (quasi) modules for quantum vertex algebras was developed in [25], where  $\phi$  is what was called an associate of the one-dimensional additive formal group (law)  $F_a(x, y) = x + y$ .

Since the very beginning, it had been recognized that the theory of vertex algebras and their modules was governed by the formal group  $F_a$ . This can be seen from the definition of a vertex algebra  $V$  and more generally the definition of a module  $(W, Y_W)$  for a given vertex algebra  $V$ , where the weak associativity axiom for a  $V$ -module  $(W, Y_W)$  states that for any  $u, v \in V$ ,  $w \in W$ , there exists a nonnegative integer  $l$  such that

$$(x + y)^l Y_W(u, x + y) Y_W(v, y) w = (x + y)^l Y_W(Y(u, x) v, y) w. \quad (1.1)$$

The notion of associate of the formal group  $F_a$  was designed in [25] to be an analog of that of  $G$ -set of a group  $G$ . By definition, an associate of  $F_a$  is a formal series  $\phi(x, z) \in \mathbb{C}((x))[[z]]$  such that

$$\phi(x, 0) = x \quad \text{and} \quad \phi(x, \phi(y, z)) = \phi(x, y + z).$$

Interestingly, it was proved therein that for any  $p(x) \in \mathbb{C}((x))$ ,  $\phi_{p(x)}(x, z) := e^{zp(x)d/dx}x$  is an associate of  $F_a$  and every associate of  $F_a$  is of this form. When  $p(x) = 1$ , we get the formal group itself, whereas when  $p(x) = x$ , we get  $\phi_{p(x)}(x, z) = xe^z$ .

Let  $\phi(x, z)$  be a general associate of  $F_a$ . The notion of  $\phi$ -coordinated (quasi)  $V$ -module for a vertex algebra (more generally for a weak quantum vertex algebra in the sense of [24])  $V$  was defined by replacing the ordinary weak associativity axiom with the property that for any  $u, v \in V$ , there is a nonnegative integer  $k$  such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

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and

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi(x_2, z)} = (\phi(x_2, z) - x_2)^k Y_W(Y(u, z)v, x_2). \quad (1.2)$$

When  $\phi(x, z) = F_a(x, z) = x + z$ , this gives an equivalent definition of the ordinary notion of module (cf. [30]). In [25], the main focus is on  $\phi$ -coordinated (quasi) modules with  $\phi(x, z) = e^{zx} \frac{d}{dx} x = xe^z$ , by which weak quantum vertex algebras were canonically associated to quantum affine algebras. Among the main results, a Jacobi-type identity and a commutator formula for  $\phi$ -coordinated modules were obtained. Later in [28],  $\phi$ -coordinated quasi modules were studied furthermore, where a commutator formula, similar to that for twisted modules (see [15]), was obtained. Just as commutator formulas for modules and twisted modules are very important and useful in the vertex algebra theory, such commutator formulas for  $\phi$ -coordinated (quasi) modules were proved to be very useful.

In this current paper, we study  $\phi_\epsilon$ -coordinated modules for vertex algebras, where  $\phi_\epsilon(x, z) = e^{zx^\epsilon} \frac{d}{dx} x$  with  $\epsilon$  an *arbitrary* integer. Part of our motivation is the fact that  $x^n \frac{d}{dx}$  with  $n \in \mathbb{Z}$  form a basis of the Witt algebra, which plays a vital role in vertex operator algebra theory and in physics conformal field theory. Conceivably,  $\phi_\epsilon$ -coordinated modules for vertex algebras will be of fundamental importance. Among the main results, we show that if  $U$  is a local subset of  $\text{Hom}(W, W((x)))$  with  $W$  a general vector space, the nonlocal vertex algebra  $\langle U \rangle_{\phi_\epsilon}$  that was obtained in [25] is a vertex algebra. We also obtain a Jacobi-type identity for  $\phi_\epsilon$ -coordinated modules for vertex algebras and furthermore we derive a commutator formula.

In this paper, we also study  $\phi_\epsilon$ -coordinated modules for some special family of vertex algebras. Note that for any Lie algebra  $\mathfrak{g}$  equipped with a symmetric invariant bilinear form  $\langle \cdot, \cdot \rangle$ , one has an (untwisted) affine Lie algebra  $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}\mathbf{c}$ . Furthermore, for every complex number  $\ell$ , one has a vertex algebra  $V_{\hat{\mathfrak{g}}}(\ell, 0)$ . This gives a very important family of vertex algebras. There is another family of vertex algebras which are associated to Novikov algebras. Recall that a (left) Novikov algebra is a non-associative algebra  $\mathcal{A}$  satisfying the condition that

$$(ab)c - a(bc) = (ba)c - b(ac), \quad (ab)c = (ac)b \quad \text{for } a, b, c \in \mathcal{A}.$$

A result of Primc (see [36]) is that for any given (left) Novikov algebra  $\mathcal{A}$  equipped with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  such that

$$\langle ab, c \rangle = \langle a, bc \rangle, \quad \langle ab, c \rangle = \langle ba, c \rangle \quad \text{for } a, b, c \in \mathcal{A},$$

one has a Lie algebra  $\tilde{L}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}\mathbf{c}$ , and furthermore, for every complex number  $\ell$  one has a vertex algebra  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ .

In this current paper, as an application of our general results we study  $\phi_\epsilon$ -coordinated modules for vertex algebras  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  associated to Novikov algebras  $\mathcal{A}$ . To determine

$\phi_\epsilon$ -coordinated modules we introduce a Lie algebra  $\widetilde{L}^\epsilon(\mathcal{A})$ , which has the same underlying space as that of  $\widetilde{L}(\mathcal{A})$ . We show that a  $\phi_\epsilon$ -coordinated module structure on a vector space for vertex algebra  $V_{\widetilde{L}(\mathcal{A})}(\ell, 0)$  exactly amounts to a “restricted” module structure for the Lie algebra  $\widetilde{L}^\epsilon(\mathcal{A})$  of level  $\ell$ .

Throughout this paper,  $\mathbb{Z}$  and  $\mathbb{N}$  denote the set of nonnegative integers and the set of integers, respectively, and all vector spaces are assumed to be over the field  $\mathbb{C}$  of complex numbers. On the other hand, we shall use the standard formal variable notations and conventions (see [15], [14], and [21]).

This paper is organized as follows: In Section 2, we recall the basic results on associates and we derive an explicit formula for  $\phi_\epsilon$ . In Section 3, we prove a Jacobi-type identity and a commutator formula for  $\phi_\epsilon$ -coordinated modules. In Section 4, we study  $\phi_\epsilon$ -coordinated modules for the vertex algebras associated to Novikov algebras.

## 2. ONE-DIMENSIONAL ADDITIVE FORMAL GROUP AND $\phi_\epsilon$ -COORDINATED MODULES FOR VERTEX ALGEBRAS

2.1. We here briefly recall the notion of formal group (cf. [17]) and the notion of associate of a formal group (see [25]).

**Definition 2.1.** A *one-dimensional formal group* over  $\mathbb{C}$  is a formal power series  $F(x, y) \in \mathbb{C}[[x, y]]$  such that

$$F(x, 0) = x, \quad F(0, y) = y, \quad F(x, F(y, z)) = F(F(x, y), z).$$

The simplest example is the one-dimensional additive formal group

$$F_a(x, y) = x + y. \tag{2.2}$$

The following notion of associate of a formal group was introduced in [25]:

**Definition 2.3.** Let  $F(x, y)$  be a one-dimensional formal group over  $\mathbb{C}$ . An *associate* of  $F(x, y)$  is a formal series  $\phi(x, z) \in \mathbb{C}((x))[[z]]$ , satisfying the condition that

$$\phi(x, 0) = x, \quad \phi(\phi(x, x_0), x_2) = \phi(x, F(x_0, x_2)).$$

The following is an explicit classification of associates for  $F_a(x, y)$  obtained in [25]:

**Proposition 2.4.** Let  $p(x) \in \mathbb{C}((x))$ . Set

$$\phi(x, z) = e^{z(p(x)d/dx)}x = \sum_{n \geq 0} \frac{z^n}{n!} \left( p(x) \frac{d}{dx} \right)^n x \in \mathbb{C}((x))[[z]].$$

Then  $\phi(x, z)$  is an associate of  $F_a(x, y)$ . Furthermore, every associate of  $F_a(x, y)$  is of this form with  $p(x)$  uniquely determined.

For any integer  $\epsilon$ , set

$$\phi_\epsilon(x, z) = e^{z(x^\epsilon d/dx)} x, \quad (2.5)$$

an associate of  $F_a(x, y)$ . For the rest of this paper, we shall be only concerned about the 1-dimensional additive formal group  $F_a(x, y)$  and its associates  $\phi_\epsilon(x, z)$ .

As special cases, we have (see [25])

$$\phi_0(x, z) = x + z, \quad \phi_1(x, z) = xe^z, \quad \phi_2(x, z) = \frac{x}{1 - zx}. \quad (2.6)$$

For the general case, from definition we have

$$\phi_{\epsilon+1}(x, z) = x + \sum_{k \geq 1} \frac{z^k}{k!} x^{k\epsilon+1} \prod_{j=0}^{k-1} (1 + j\epsilon). \quad (2.7)$$

Assume  $\epsilon \neq 0$ . For  $k \geq 1$ , we have

$$\frac{1}{k!} \prod_{j=0}^{k-1} (1 + j\epsilon) = (-\epsilon)^k \frac{1}{k!} \prod_{j=0}^{k-1} \left(-\frac{1}{\epsilon} - j\right) = (-\epsilon)^k \binom{-\frac{1}{\epsilon}}{k}.$$

Then

$$\phi_{\epsilon+1}(x, z) = x + x \sum_{k \geq 1} (-\epsilon)^k \binom{-\frac{1}{\epsilon}}{k} (zx^\epsilon)^k = x(1 - \epsilon zx^\epsilon)^{-\frac{1}{\epsilon}}. \quad (2.8)$$

Notice that

$$\lim_{\epsilon \rightarrow 0} \phi_{\epsilon+1}(x, z) = xe^z = \phi_1(x, z). \quad (2.9)$$

The following are some simple facts we shall use:

**Lemma 2.10.** *We have*

$$\phi_\epsilon(x, z) - x = zh(x, z), \quad (2.11)$$

$$\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2) = (x_1 - x_2)g(x, x_1, x_2), \quad (2.12)$$

where  $h(x, z)$  is a unit in  $\mathbb{C}((x))[[z]]$  and  $g(x, x_1, x_2)$  is a unit in  $\mathbb{C}((x))[[x_1, x_2]]$ .

*Proof.* By definition we have

$$\phi_\epsilon(x, z) - x = z \sum_{j \geq 1} \frac{1}{j!} z^{j-1} \left(x^\epsilon \frac{d}{dx}\right)^j x.$$

Set  $h(x, z) = \sum_{j \geq 1} \frac{1}{j!} z^{j-1} \left(x^\epsilon \frac{d}{dx}\right)^j x \in \mathbb{C}((x))[[z]]$ . As  $h(x, 0) = 1$ ,  $h(x, z)$  is a unit in  $\mathbb{C}((x))[[z]]$ . On the other hand, by definition we have

$$\begin{aligned} \phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2) &= \left(e^{x_1 x^\epsilon \frac{d}{dx}} - e^{x_2 x^\epsilon \frac{d}{dx}}\right) x = \left(e^{(x_1 - x_2) x^\epsilon \frac{d}{dx}} - 1\right) e^{x_2 x^\epsilon \frac{d}{dx}} \cdot x \\ &= (x_1 - x_2) \sum_{j \geq 1} \frac{1}{j!} (x_1 - x_2)^{j-1} \left(x^\epsilon \frac{d}{dx}\right)^j \phi_\epsilon(x, x_2). \end{aligned}$$

Set

$$g(x, x_1, x_2) = \sum_{j \geq 1} \frac{1}{j!} (x_1 - x_2)^{j-1} \left( x^\epsilon \frac{d}{dx} \right)^j \phi_\epsilon(x, x_2).$$

We have  $g(x, x_1, x_2) \in \mathbb{C}((x))[[x_1, x_2]]$  and  $g(x, x_2, x_2) = \phi_\epsilon(x, x_2)$ . Note that  $\phi_\epsilon(x, x_2)$  is a unit in  $\mathbb{C}((x))[[x_2]]$  as  $\phi_\epsilon(x, 0) = x$  (nonzero in  $\mathbb{C}((x))$ ). It then follows that  $g(x, x_1, x_2)$  is a unit in  $(\mathbb{C}((x))[[x_2]])[[x_1]] = \mathbb{C}((x))[[x_1, x_2]]$ .  $\square$

2.2. Next we recall the definition of a (nonlocal) vertex algebra and the definitions of a module and a  $\phi_\epsilon$ -coordinated module for a (nonlocal) vertex algebra. The notion of nonlocal vertex algebra was studied in [23] (under the name “axiomatic  $G_1$ -vertex algebra”) and in [24], and it was also independently studied in [4] (under the name “field algebra”). The theory of  $\phi$ -coordinated modules for quantum vertex algebras was developed in [26], whereas  $\phi_1$ -coordinated modules were the main focus therein.

**Definition 2.13.** A *nonlocal vertex algebra* is a vector space  $V$  equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End}V) \end{aligned}$$

and with a distinguished vector  $\mathbf{1} \in V$ , satisfying the conditions that

$$Y(\mathbf{1}, x) = 1, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V$$

and that for any  $u, v, w \in V$ , there exists a nonnegative integer  $l$  such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w. \quad (2.14)$$

Furthermore, a *vertex algebra* is a nonlocal vertex algebra  $V$  satisfying the condition that for any  $u, v \in V$ , there exists a nonnegative integer  $k$  such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k Y(v, x_2) Y(u, x_1). \quad (2.15)$$

For each nonlocal vertex algebra  $V$ , there is a canonical operator  $\mathcal{D}$  on  $V$ , which is defined by

$$\mathcal{D}(v) = v_{-2}\mathbf{1} = \left( \frac{d}{dx} Y(v, x)\mathbf{1} \right) \Big|_{x=0} \quad \text{for } v \in V.$$

This operator  $\mathcal{D}$  satisfies the following property:

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x). \quad (2.16)$$

**Definition 2.17.** Let  $V$  be a vertex algebra. A  $V$ -module is a vector space  $W$  equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : V &\rightarrow \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]], \\ v &\mapsto Y_W(v, x), \end{aligned}$$

satisfying the conditions that  $Y_W(1, x) = 1_W$  (the identity operator on  $W$ ) and that for  $u, v \in V$ ,  $w \in W$ , there exists  $l \in \mathbb{N}$  such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \quad (2.18)$$

**Remark 2.19.** It was shown in [30] (Lemma 2.9) that the weak associativity axiom in the definition of a  $V$ -module can be equivalently replaced by the condition that for any  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (2.20)$$

$$x_0^k Y_W(Y(u, x_0)v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=x_2+x_0}. \quad (2.21)$$

**Definition 2.22.** Let  $V$  be a nonlocal vertex algebra and let  $\phi$  be an associate of the one-dimensional additive formal group  $F_a$ . A  $\phi$ -coordinated  $V$ -module is a vector space  $W$  equipped with a linear map  $Y_W(\cdot, x)$  as in Definition 2.17, satisfying the conditions that  $Y_W(1, x) = 1_W$  and that for  $u, v \in V$ , there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (2.23)$$

$$\begin{aligned} &(\phi(x_2, x_0) - x_2)^k Y_W(Y(u, x_0)v, x_2) \\ &= ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}. \end{aligned} \quad (2.24)$$

It is clear that the notion of  $\phi_0$ -coordinated  $V$ -module is equivalent to that of  $V$ -module.

### 3. JACOBI-TYPE IDENTITY FOR $\phi_\epsilon$ -COORDINATED MODULES

In this section, we shall present some axiomatic results on  $\phi_\epsilon$ -coordinated modules for vertex algebras. In particular, we establish a Jacobi-type identity and a commutator formula.

Let  $W$  be a vector space. Set

$$\mathcal{E}(W) = \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]]. \quad (3.1)$$

A subset  $U$  of  $\mathcal{E}(W)$  is said to be *local* if for any  $a(x), b(x) \in U$  there exists a nonnegative integer  $k$  such that

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1). \quad (3.2)$$

A pair  $(a(x), b(x))$  in  $\mathcal{E}(W)$  is said to be *compatible* if there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (3.3)$$

Note that (3.2) implies (3.3). Thus each pair in a local subset is always compatible.

Fix an integer  $\epsilon$  throughout this section. Let  $(a(x), b(x))$  be any compatible pair in  $\mathcal{E}(W)$  with  $k \in \mathbb{N}$  such that (3.3) holds. We define  $a(x)_n^\epsilon b(x) \in \mathcal{E}(W)$  for  $n \in \mathbb{Z}$  in terms of generating function

$$Y_{\mathcal{E}}^\epsilon(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^\epsilon b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}^\epsilon(a(x), z)b(x) = (\phi_\epsilon(x, z) - x)^{-k} ((x_1 - x)^k a(x_1)b(x))|_{x_1=\phi_\epsilon(x, z)}, \quad (3.4)$$

where  $(\phi_\epsilon(x, z) - x)^{-k}$  is viewed as an element of  $\mathbb{C}((x))((z))$  (recalling Lemma 2.10).

**Remark 3.5.** A notion of compatible subset of  $\mathcal{E}(W)$  was introduced in [23] and it was proved therein that each compatible subset generates a nonlocal vertex algebra in a certain canonical way. It was proved in [25] (Theorem 4.10) that any compatible subset  $U$  of  $\mathcal{E}(W)$  generates a nonlocal vertex algebra  $\langle U \rangle_{\phi_\epsilon}$  with  $W$  as a canonical  $\phi_\epsilon$ -coordinated module. On the other hand, it was proved in [23] that every local subset of  $\mathcal{E}(W)$  is compatible. Thus each local subset  $U$  of  $\mathcal{E}(W)$  generates a nonlocal vertex algebra  $\langle U \rangle_{\phi_\epsilon}$ . It was proved in [25] that  $\langle U \rangle_{\phi_1}$  is a vertex algebra, whereas it was proved in [22] that  $\langle U \rangle_{\phi_0}$  is a vertex algebra.

In the following, we shall prove that for every integer  $\epsilon$ ,  $\langle U \rangle_{\phi_\epsilon}$  is a vertex algebra, generalizing the corresponding results of [22, 25].

**Proposition 3.6.** *Let  $W$  be a vector space and let  $V$  be a local subspace of  $\mathcal{E}(W)$ , which is  $Y_{\mathcal{E}}^\epsilon$ -closed in the sense that*

$$u(x)_n^\epsilon v(x) \in V \quad \text{for } u(x), v(x) \in V, \quad n \in \mathbb{Z}.$$

*Let  $a(x), b(x) \in V$ . Suppose*

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1) \quad (3.7)$$

*for some nonnegative integer  $k$ . Then*

$$(x_1 - x_2)^k Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) = (x_1 - x_2)^k Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1). \quad (3.8)$$

*Proof.* Let  $c(x) \in V$  be arbitrarily fixed. There exists  $l \in \mathbb{N}$  with  $l \geq k$  such that

$$(z - x)^l a(z)c(x) = (z - x)^l c(x)a(z), \quad (z - x)^l b(z)c(x) = (z - x)^l c(x)b(z).$$

Using this and (3.7) we get

$$(y - z)^l (y - x)^l (z - x)^l a(y)b(z)c(x) \in \text{Hom}(W, W((x, y, z))).$$

By Lemma 4.7 in [25], we have

$$\begin{aligned} & (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^l (\phi_\epsilon(x, x_1) - x)^l (\phi_\epsilon(x, x_2) - x)^l Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) c(x) \\ &= (y - z)^l (y - x)^l (z - x)^l a(y) b(z) c(x) |_{y=\phi_\epsilon(x, x_1), z=\phi_\epsilon(x, x_2)}. \end{aligned}$$

Set

$$f(x, x_1, x_2) = (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^l (\phi_\epsilon(x, x_1) - x)^l (\phi_\epsilon(x, x_2) - x)^l,$$

which lies in  $\mathbb{C}((x))((x_1, x_2))$ . Then

$$\begin{aligned} & f(x, x_1, x_2) (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^k Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) c(x) \\ &= (y - z)^l (y - x)^l (z - x)^l (y - z)^k a(y) b(z) c(x) |_{y=\phi_\epsilon(x, x_1), z=\phi_\epsilon(x, x_2)} \\ &= (y - z)^l (y - x)^l (z - x)^l (y - z)^k b(z) a(y) c(x) |_{z=\phi_\epsilon(x, x_2), y=\phi_\epsilon(x, x_1)} \\ &= f(x, x_1, x_2) (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^k Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1) c(x). \end{aligned}$$

Noticing that  $(\phi_\epsilon(x, x_1) - x)^l (\phi_\epsilon(x, x_2) - x)^l$  is invertible in  $\mathbb{C}((x))((x_1, x_2))$ , by cancellation, we get

$$\begin{aligned} & (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^{l+k} Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) c(x) \\ &= (\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^{l+k} Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1) c(x). \end{aligned} \quad (3.9)$$

By Lemma 2.10, we have

$$(\phi_\epsilon(x, x_1) - \phi_\epsilon(x, x_2))^{l+k} = (x_1 - x_2)^{l+k} g(x, x_1, x_2)^{l+k},$$

where  $g(x, x_1, x_2)$  is a unit in  $\mathbb{C}((x))[[x_1, x_2]]$ . By cancellation we get

$$(x_1 - x_2)^{l+k} Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) = (x_1 - x_2)^{l+k} Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1). \quad (3.10)$$

Combining this with the weak associativity obtained in [25] we get

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1) \\ &= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_{\mathcal{E}}^\epsilon(Y_{\mathcal{E}}^\epsilon(a(x), x_0) b(x), x_2). \end{aligned} \quad (3.11)$$

From (3.7) we have

$$(x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))),$$

so that

$$(\phi_\epsilon(x_2, x_0) - x_2)^k Y_{\mathcal{E}}^\epsilon(a(x), x_0) b(x) = (x_1 - x)^k a(x_1) b(x) |_{x_1=\phi_\epsilon(x_2, x_0)}.$$

Multiplying both sides of (3.11) by  $(\phi_\epsilon(x_2, x_0) - x_2)^k$  and then taking  $\text{Res}_{x_0}$  we get

$$\begin{aligned} & (\phi_\epsilon(x_2, x_0) - x_2)^k Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) \\ &= (\phi_\epsilon(x_2, x_0) - x_2)^k Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1). \end{aligned}$$



By Lemma 2.10, we have

$$(\phi_\epsilon(x_2, x_0) - x_2)^k = x_0^k h(x_2, x_0)^k$$

where  $h(x_2, x_0)$  is a unit in  $\mathbb{C}((x_2))[[x_0]]$ . By cancellation we obtain

$$(x_1 - x_2)^k Y_{\mathcal{E}}^\epsilon(a(x), x_1) Y_{\mathcal{E}}^\epsilon(b(x), x_2) = (x_1 - x_2)^k Y_{\mathcal{E}}^\epsilon(b(x), x_2) Y_{\mathcal{E}}^\epsilon(a(x), x_1),$$

as desired.  $\square$

Now we have:

**Theorem 3.12.** *Let  $W$  be a vector space and let  $U$  be any local subset of  $\mathcal{E}(W)$ . Then  $\langle U \rangle_{\phi_\epsilon}$  is a vertex algebra and  $W$  is a canonical  $\phi_\epsilon$ -coordinated  $\langle U \rangle_{\phi_\epsilon}$ -module.*

*Proof.* We already knew that  $\langle U \rangle_{\phi_\epsilon}$  is a nonlocal vertex algebra and  $W$  is a  $\phi_\epsilon$ -coordinated  $\langle U \rangle_{\phi_\epsilon}$ -module with  $Y_W(\alpha(x), z) = \alpha(z)$  for  $\alpha(x) \in \langle U \rangle_{\phi_\epsilon}$ . As  $\langle U \rangle_{\phi_\epsilon}$  is the smallest  $Y_{\mathcal{E}}^\epsilon$ -closed local subspace containing  $U$  and  $1_W$ , we see that  $\langle U \rangle_{\phi_\epsilon}$  as a nonlocal vertex algebra is generated by  $U$ . Given that  $U$  is local, by Proposition 3.6 we see that

$$\{Y_{\mathcal{E}}^\epsilon(a(x), z) \mid a(x) \in U\}$$

is a local subset of  $\mathcal{E}(\langle U \rangle_{\phi_\epsilon})$ . It follows that  $\langle U \rangle_{\phi_\epsilon}$  is a vertex algebra and  $W$  is a  $\phi_\epsilon$ -coordinated  $\langle U \rangle_{\phi_\epsilon}$ -module.  $\square$

We also have the following results:

**Proposition 3.13.** *Let  $V$  be a vertex algebra and let  $(W, Y_W)$  be a  $\phi_\epsilon$ -coordinated  $V$ -module. Suppose that for some fixed  $u, v \in V$ ,  $k \in \mathbb{N}$ ,*

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

*Then*

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1). \quad (3.14)$$

*Proof.* Recall the skew-symmetry of  $V$ :

$$Y(u, x)v = e^{x\mathcal{D}}Y(v, -x)u \quad \text{for } u, v \in V.$$

From the definition, there exists  $l \in \mathbb{N}$  such that

$$(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2), \quad (x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1) \in \text{Hom}(W, W((x_1, x_2))).$$

Then, using Lemmas 3.6 and 3.7 in [25] we get

$$\begin{aligned} & ((x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi_\epsilon(x_2, x_0)} \\ &= (\phi_\epsilon(x_2, x_0) - x_2)^l Y_W(Y(u, x_0)v, x_2) \\ &= (\phi_\epsilon(x_2, x_0) - x_2)^l Y_W(e^{x_0\mathcal{D}}Y(v, -x_0)u, x_2) \\ &= (\phi_\epsilon(x_2, x_0) - x_2)^l Y_W(Y(v, -x_0)u, \phi_\epsilon(x_2, x_0)). \end{aligned}$$

On the other hand, we have

$$((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_2=\phi_\epsilon(x_1, -x_0)} = (x_1 - \phi_\epsilon(x_1, -x_0))^l Y_W(Y(v, -x_0)u, x_1).$$

Hence

$$\begin{aligned} & ((x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi_\epsilon(x_2, x_0)} \\ &= (((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_2=\phi_\epsilon(x_1, -x_0)})|_{x_1=\phi_\epsilon(x_2, x_0)} \\ &= ((x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1))|_{x_1=\phi_\epsilon(x_2, x_0)}. \end{aligned}$$

It follows that

$$(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^l Y_W(v, x_2) Y_W(u, x_1),$$

which implies

$$(x_1 - x_2)^l (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^l (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1).$$

Noticing that  $(x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1) \in \text{Hom}(W, W((x_2))((x_1)))$  and

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \subset \text{Hom}(W, W((x_2))((x_1)))$$

by assumption, we can multiply both sides by the inverse of  $(x_1 - x_2)^l$  in  $\mathbb{C}((x_2))((x_1))$  to obtain the desired relation (3.14).  $\square$

**Proposition 3.15.** *Let  $V$  be a vertex algebra and let  $(W, Y_W)$  be a  $\phi_\epsilon$ -coordinated  $V$ -module. Set  $V_W = \{Y_W(v, x) | v \in V\}$ . Then  $V_W$  is a local subspace of  $\mathcal{E}(W)$ ,  $(V_W, Y_\mathcal{E}^\epsilon, 1_W)$  is a vertex algebra, and  $Y_W$  is a homomorphism of vertex algebras.*

*Proof.* By Definition 2.22 and Proposition 3.13,  $V_W$  is a local subspace of  $\mathcal{E}(W)$ . Let  $u, v \in V$ . By definition, there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x)^k Y_W(u, x_1) Y_W(v, x) \in \text{Hom}(W, W((x_1, x)))$$

and

$$(\phi_\epsilon(x_2, z) - x_2)^k Y_\mathcal{E}^\epsilon(Y(u, x), z) Y_W(v, x) = (x_1 - x)^k Y_W(u, x_1) Y_W(v, x)|_{x_1=\phi_\epsilon(x_2, z)}.$$

On the other hand, from the definition of  $Y_\mathcal{E}^\epsilon(\cdot, x)$  we have

$$(\phi_\epsilon(x_2, z) - x_2)^k Y_\mathcal{E}^\epsilon(Y_W(u, x), z) Y_W(v, x) = (x_1 - x)^k Y_W(u, x_1) Y_W(v, x)|_{x_1=\phi_\epsilon(x_2, z)}.$$

It follows that

$$(\phi_\epsilon(x_2, z) - x_2)^k Y_\mathcal{E}^\epsilon(Y(u, x), z) Y_W(v, x) = (\phi_\epsilon(x_2, z) - x_2)^k Y_\mathcal{E}^\epsilon(Y_W(u, x), z) Y_W(v, x).$$

Since both  $Y_\mathcal{E}^\epsilon(Y(u, x), z) Y_W(v, x)$  and  $Y_\mathcal{E}^\epsilon(Y_W(u, x), z) Y_W(v, x)$  involve only finitely many negative powers of  $z$ , and since  $(\phi_\epsilon(x_2, z) - x_2)^k$  is a unit in  $\mathbb{C}((x_2))((z))$  (by Lemma 2.10), by cancellation we get

$$Y_\mathcal{E}^\epsilon(Y(u, x), z) Y_W(v, x) = Y_\mathcal{E}^\epsilon(Y_W(u, x), z) Y_W(v, x).$$

Then  $(V_W, Y_{\mathcal{E}}^{\epsilon}, 1_W)$  is a vertex algebra, and  $Y_W$  is a homomorphism of vertex algebras.  $\square$

We also have the following result generalizing the corresponding results of [23, 25]:

**Lemma 3.16.** *Let  $W$  be a vector space and let*

$$\begin{aligned} A(x_1, x_2) &\in \text{Hom}(W, W((x_1))((x_2))), & B(x_1, x_2) &\in \text{Hom}(W, W((x_2))((x_1))), \\ C(x_0, x_2) &\in (\text{Hom}(W, W((x_2))))((x_0)). \end{aligned}$$

*If there exists a nonnegative integer  $k$  such that*

$$\begin{aligned} (x_1 - x_2)^k A(x_1, x_2) &= (x_1 - x_2)^k B(x_1, x_2), \\ ((x_1 - x_2)^k A(x_1, x_2))|_{x_1=\phi_{\epsilon}(x_2, x_0)} &= (\phi_{\epsilon}(x_2, x_0) - x_2)^k C(x_0, x_2), \end{aligned}$$

*then*

$$\begin{aligned} (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) A(x_1, x_2) &- (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) B(x_1, x_2) \\ &= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) C(f_{\epsilon}(x_2, z), x_2), \end{aligned} \quad (3.17)$$

*where*

$$f_{\epsilon}(x_2, z) = \begin{cases} x_2^{1-\epsilon} \cdot \frac{(1+z)^{1-\epsilon}-1}{1-\epsilon}, & \text{for } \epsilon \neq 1, \\ \log(1+z), & \text{for } \epsilon = 1. \end{cases}$$

*Proof.* We start with the standard delta-function identity

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right).$$

Substituting  $x_0 = x_2 z$  with  $z$  a new formal variable, we have

$$(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) = x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right). \quad (3.18)$$

Then we get

$$\begin{aligned}
& (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_2 z)^k A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_2 z)^k B(x_1, x_2) \\
&= (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_1 - x_2)^k A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_1 - x_2)^k B(x_1, x_2) \\
&= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2)) \\
&= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2)) \big|_{x_1=x_2(1+z)} \\
&= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) (((x_1 - x_2)^k A(x_1, x_2)) \big|_{x_1=\phi_\epsilon(x_2, x_0)}) \big|_{x_0=f_\epsilon(x_2, z)} \\
&= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) (x_2 z)^k C(f_\epsilon(x_2, z), x_2),
\end{aligned}$$

noticing that

$$\phi_\epsilon(x, f_\epsilon(x, z)) = x(1+z).$$

Then (3.17) follows.  $\square$

As the main result of this section we have:

**Theorem 3.19.** *Let  $V$  be a vertex algebra and let  $(W, Y_W)$  be a  $\phi_\epsilon$ -coordinated module. Then*

$$\begin{aligned}
& (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) Y_W(u, x_1) Y_W(v, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) Y_W(v, x_2) Y_W(u, x_1) \\
&= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) Y_W(Y(u, f_\epsilon(x_2, z))v, x_2)
\end{aligned} \tag{3.20}$$

for  $u, v \in V$ , where

$$f_\epsilon(x, z) = \begin{cases} x^{1-\epsilon} \cdot \frac{(1+z)^{1-\epsilon}-1}{1-\epsilon}, & \text{for } \epsilon \neq 1, \\ \log(1+z), & \text{for } \epsilon = 1. \end{cases}$$

Furthermore, we have

$$[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{j \geq 0} \frac{1}{j!} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) Y_W(u_j v, x_2). \tag{3.21}$$

*Proof.* From definition, there exists  $k \in \mathbb{N}$  such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$(\phi_\epsilon(x_2, x_0) - x_2)^k Y_W(Y(u, x_0)v, x_2) = ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1=\phi_\epsilon(x_2, x_0)}.$$

On the other hand, by Proposition 3.13 we also have

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1). \quad (3.22)$$

Then the first assertion follows immediately from Lemma 3.16. Furthermore, applying  $\text{Res}_z x_2$  we get

$$\begin{aligned} & [Y_W(u, x_1), Y_W(v, x_2)] \\ = & \text{Res}_z x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) x_2 Y_W(Y(u, f_\epsilon(x_2, z))v, x_2) \\ = & \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) x_2 \frac{\partial}{\partial x_0} \left( \frac{\phi_\epsilon(x_2, x_0)}{x_2} - 1 \right) Y_W(Y(u, x_0)v, x_2) \\ = & \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) \phi_\epsilon(x_2, x_0)^\epsilon Y_W(Y(u, x_0)v, x_2) \\ = & \text{Res}_{x_0} x_1^{\epsilon-1} \delta \left( \frac{\phi_\epsilon(x_2, x_0)}{x_1} \right) Y_W(Y(u, x_0)v, x_2) \\ = & \sum_{j \geq 0} \frac{1}{j!} \left[ \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) \right] Y_W(u_j v, x_2), \end{aligned}$$

noticing that

$$\frac{\partial}{\partial x_0} \phi_\epsilon(x_2, x_0) = e^{x_0 x_2^\epsilon \frac{\partial}{\partial x_2}} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) (x_2) = e^{x_0 x_2^\epsilon \frac{\partial}{\partial x_2}} (x_2^\epsilon) = \left( e^{x_0 x_2^\epsilon \frac{\partial}{\partial x_2}} x_2 \right)^\epsilon = \phi_\epsilon(x_2, x_0)^\epsilon.$$

This proves the second assertion.  $\square$

**Remark 3.23.** We here collect some basic facts that we shall use. We have

$$\left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) = - \left( x_1^\epsilon \frac{\partial}{\partial x_1} \right) x_2^{\epsilon-1} \delta \left( \frac{x_1}{x_2} \right), \quad (3.24)$$

$$(x_1 - x_2)^m \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = 0 \quad (3.25)$$

for any nonnegative integers  $m$  and  $n$  with  $m > n$ , and

$$(x_1 - x_2)^n \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = x_2^{n\epsilon} (x_1 - x_2)^n \left( \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) \quad (3.26)$$

for any nonnegative integer  $n$ . Furthermore, we have

$$\text{Res}_{x_1} x_1^{-\epsilon} (x_1 - x_2)^n \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) = \frac{x_2^{n\epsilon}}{n!}. \quad (3.27)$$

These facts can be proved by using the special case with  $\epsilon = 0$  (cf. [22]) and the facts that for any positive integer  $n$ , there exists polynomials  $f_1(x), \dots, f_n(x)$  such that

$$\left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^n = x_2^{n\epsilon} \left( \frac{\partial}{\partial x_2} \right)^n + f_1(x_2) \left( \frac{\partial}{\partial x_2} \right)^{n-1} + \dots + f_n(x_2). \quad (3.28)$$

The following, which is a generalization of a result in [22], follows immediately from Theorem 3.19 and the basic facts in Remark 3.23:

**Lemma 3.29.** *Let  $V$  be a vertex algebra and let  $(W, Y_W)$  be a faithful  $\phi_\epsilon$ -coordinated  $V$ -module. Suppose that*

$$[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{j \geq 0} \frac{1}{j!} \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^j x_1^{\epsilon-1} \delta \left( \frac{x_2}{x_1} \right) Y_W(A^j, x_2), \quad (3.30)$$

where  $u, v, A^0, A^1, \dots$  are fixed vectors in  $V$ . Then  $A^j = u_j v$  for all  $j \geq 0$ .

#### 4. VERTEX ALGEBRAS ARISING FROM NOVIKOV ALGEBRAS

In this section, we study  $\phi_\epsilon$ -coordinated modules for the vertex algebras associated to Novikov algebras by Primc.

We first recall the definition of a Novikov algebra (see [6], [16], [32]).

**Definition 4.1.** A (left) Novikov algebra is a non-associative algebra  $\mathcal{A}$  satisfying

$$(ab)c - a(bc) = (ba)c - b(ac), \quad (4.2)$$

$$(ab)c = (ac)b \quad (4.3)$$

for  $a, b, c \in \mathcal{A}$ .

Note that any commutative and associative algebra is a Novikov algebra.

**Remark 4.4.** We here recall the Gelfand construction of Novikov algebras due to S. Gelfand (see [16]). Let  $A$  be a commutative associative algebra with a derivation  $\partial$ . Define a new operation  $\circ$  on  $A$  by  $a \circ b = a\partial b$  for  $a, b \in A$ . Then  $(A, \circ)$  is a (left) Novikov algebra.

The following result was due to Balinsky and Novikov (see [6]):

**Proposition 4.5.** *Let  $\mathcal{A}$  be a non-associative algebra. Set*

$$L(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}], \quad \partial = \frac{d}{dt}. \quad (4.6)$$

Define a bilinear operation  $[\cdot, \cdot]$  on  $L(\mathcal{A})$  by

$$[a \otimes f, b \otimes g] = ab \otimes (\partial f)g - ba \otimes (\partial g)f \quad (4.7)$$

for  $a, b \in \mathcal{A}$ ,  $f, g \in \mathbb{C}[t, t^{-1}]$ . Then  $(L(\mathcal{A}), [\cdot, \cdot])$  is a Lie algebra if and only if  $\mathcal{A}$  is a Novikov algebra.

The following refinement was due to Primc (see [36], Example 3; cf. [6]):

**Proposition 4.8.** *Let  $\mathcal{A}$  be a non-associative algebra equipped with a bilinear form  $\langle \cdot, \cdot \rangle$ . Set*

$$\tilde{L}(\mathcal{A}) = L(\mathcal{A}) \oplus \mathbb{C}\mathbf{c},$$

*where  $\mathbf{c}$  is a distinguished nonzero element. For  $a \in \mathcal{A}$ ,  $m \in \mathbb{Z}$ , set  $L(a, m) = a \otimes t^{m+1}$ . Define a bilinear operation  $[\cdot, \cdot]$  on  $\tilde{L}(\mathcal{A})$  by*

$$\begin{aligned} [L(a, m), L(b, n)] &= (m+1)L(ab, m+n) - (n+1)L(ba, m+n) \\ &\quad + \frac{1}{12}(m^3 - m)\langle a, b \rangle \delta_{m+n, 0} \mathbf{c}, \end{aligned} \quad (4.9)$$

$$[\mathbf{c}, \tilde{L}(\mathcal{A})] = 0 = [\tilde{L}(\mathcal{A}), \mathbf{c}] \quad (4.10)$$

*for  $a, b \in \mathcal{A}$ ,  $m, n \in \mathbb{Z}$ . Then  $(\tilde{L}(\mathcal{A}), [\cdot, \cdot])$  is a Lie algebra if and only if  $\mathcal{A}$  is a Novikov algebra and  $\langle \cdot, \cdot \rangle$  is a symmetric form satisfying*

$$\langle ab, c \rangle = \langle a, bc \rangle, \quad \langle ab, c \rangle = \langle ba, c \rangle \quad \text{for } a, b, c \in \mathcal{A}. \quad (4.11)$$

**Remark 4.12.** Note that a unital Novikov algebra  $\mathcal{A}$  with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  satisfying (4.11) amounts to a Frobenius algebra, i.e., a unital commutative and associative algebra with a nondegenerate symmetric and associative form. For a Frobenius algebra  $\mathcal{A}$ , the corresponding Lie algebra  $\tilde{L}(\mathcal{A})$  is isomorphic to the map Virasoro algebra (see [20, 31]). In particular, if  $\mathcal{A} = \mathbb{C}e$  is 1-dimensional with  $e \cdot e = e$  and  $\langle e, e \rangle = \frac{1}{12}$ , then  $\tilde{L}(\mathcal{A})$  is isomorphic to the Virasoro algebra. More examples can be found in [33, 34, 35].

**Example 4.13.** Let  $\mathcal{A} = \mathbb{C}[x, x^{-1}]$  and let  $p(x) \in \mathbb{C}[x, x^{-1}]$ . By the Gelfand construction, one has a Novikov algebra  $(\mathcal{A}, \circ_{p(x)})$ , where

$$x^i \circ_{p(x)} x^j = x^i \left( p(x) \frac{d}{dx} \right) x^j = j x^{i+j-1} p(x) \quad \text{for } i, j \in \mathbb{Z}.$$

In this case, the corresponding Lie algebra  $L(\mathcal{A})$  is a Lie algebra of Block type (cf. [7, 10]). In particular, if  $p(x) = 1$ ,  $L(\mathcal{A})$  is isomorphic to the Poisson algebra  $\mathbb{C}[x, x^{-1}, y, y^{-1}]$  with bracket relation

$$[f, g] = \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \quad \text{for } f, g \in \mathbb{C}[x, x^{-1}, y, y^{-1}].$$

For this special case, taking a basis  $L_m^i = x^{i+1+m} y^{m+1}$  for  $m, i \in \mathbb{Z}$ , we have

$$[L_m^i, L_n^j] = (j(m+1) - i(n+1)) L_{m+n}^{i+j} \quad \text{for } i, j, m, n \in \mathbb{Z}. \quad (4.14)$$

The structure and representation theory of this Lie algebra and its subalgebras have been extensively studied in [3, 37, 38, 39].

Let  $\mathcal{A}$  be a Novikov algebra equipped with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  satisfying (4.11). Following Primc [36] we associate vertex algebras to  $\mathcal{A}$ . For  $a \in \mathcal{A}$ , set

$$L(a, x) = \sum_{n \in \mathbb{Z}} L(a, n) x^{-n-2} \in \tilde{L}(\mathcal{A})[[x, x^{-1}]]. \quad (4.15)$$

In terms of generating functions the relation (4.9) can be rewritten as

$$\begin{aligned}
& [L(a, x_1), L(b, x_2)] \\
&= \left( \frac{\partial}{\partial x_2} L(ba, x_2) \right) x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) + (L(ab, x_2) + L(ba, x_2)) \left( \frac{\partial}{\partial x_2} \right) x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) \\
&\quad + \frac{1}{12} \langle a, b \rangle \mathbf{c} \left( \frac{\partial}{\partial x_2} \right)^3 x_1^{-1} \delta \left( \frac{x_2}{x_1} \right)
\end{aligned} \tag{4.16}$$

for  $a, b \in \mathcal{A}$ . Set

$$\tilde{L}(\mathcal{A})_+ = \mathcal{A} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{c}, \quad \tilde{L}(\mathcal{A})_- = \mathcal{A} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Note that  $\tilde{L}(\mathcal{A})_{\pm}$  are Lie subalgebras and  $\tilde{L}(\mathcal{A}) = \tilde{L}(\mathcal{A})_+ \oplus \tilde{L}(\mathcal{A})_-$  as a vector space. Let  $\ell \in \mathbb{C}$  and denote by  $\mathbb{C}_{\ell}$  the one-dimensional  $\tilde{L}(\mathcal{A})_+$ -module with  $\mathbf{c}$  acting as scalar  $\ell$  and with  $\mathcal{A} \otimes \mathbb{C}[t]$  acting trivially. Form an induced module

$$V_{\tilde{L}(\mathcal{A})}(\ell, 0) = U(\tilde{L}(\mathcal{A})) \otimes_{U(\tilde{L}(\mathcal{A})_+)} \mathbb{C}_{\ell}. \tag{4.17}$$

Set  $\mathbf{1} = 1 \otimes 1 \in V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  and identify  $\mathcal{A}$  as a subspace of  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  through the linear map

$$a \mapsto L(a, -2)\mathbf{1} \quad \text{for } a \in \mathcal{A}.$$

From [36] (cf. [11, 40]), there exists a vertex algebra structure on  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ , which is uniquely determined by the condition that  $\mathbf{1}$  is the vacuum vector and

$$Y(a, x) = L(a, x) = \sum_{n \in \mathbb{Z}} L(a, n) x^{-n-2} \quad \text{for } a \in \mathcal{A}.$$

Furthermore,  $\mathcal{A}$  is a generating subspace of vertex algebra  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  with

$$a_0 b = \mathcal{D}(ba), \quad a_1 b = ab + ba, \quad a_3 b = \frac{1}{2} \ell \langle a, b \rangle \mathbf{1}, \quad a_2 b = 0 = a_k b \tag{4.18}$$

for  $a, b \in \mathcal{A}$  and for  $k \geq 4$ .

Next, we discuss a  $\mathbb{Z}$ -graded vertex algebra structure on  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ .

**Definition 4.19.** A  $\mathbb{Z}$ -graded vertex algebra is a vertex algebra  $V$  equipped with a  $\mathbb{Z}$ -grading  $V = \oplus_{n \in \mathbb{Z}} V_n$  such that  $\mathbf{1} \in V_0$  and

$$u_k v \in V_{m+n-k-1} \quad \text{for } u \in V_m, v \in V_n, m, n, k \in \mathbb{Z}. \tag{4.20}$$

Let  $(\mathcal{A}, \langle \cdot, \cdot \rangle)$  be given as before. It can be readily seen that  $\tilde{L}(\mathcal{A})$  is a  $\mathbb{Z}$ -graded Lie algebra with  $\deg \mathbf{c} = 0$  and

$$\deg(a \otimes t^m) = \deg(L(a, m-1)) = -m+1 \quad \text{for } a \in \mathcal{A}, m \in \mathbb{Z}. \tag{4.21}$$

As  $\tilde{L}(\mathcal{A})_+$  is a graded subalgebra,  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  is naturally a  $\mathbb{Z}$ -graded  $\tilde{L}(\mathcal{A})$ -module with  $\deg \mathbf{1} = 0$  and with  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)_2 = \mathcal{A}$ . Furthermore, by Lemma A (in Appendix)  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$



equipped with this  $\mathbb{Z}$ -grading is a  $\mathbb{Z}$ -graded vertex algebra. In view of the PBW Theorem,  $\mathcal{D}a = a_{-2}\mathbf{1} \neq 0$  for any nonzero  $a \in \mathcal{A}$  and  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  is linearly spanned by the vectors

$$a_{-m_1}^{(1)} \cdots a_{-m_r}^{(r)} \mathbf{1}$$

for  $r \geq 0$ ,  $a^{(i)} \in \mathcal{A}$ ,  $m_i \geq 1$ .

On the other hand, we have (also see [2]):

**Proposition 4.22.** *Let  $V = \oplus_{n \in \mathbb{Z}} V_n$  be a  $\mathbb{Z}$ -graded vertex algebra with the following properties:*

- (1)  $V_n = 0$  for  $n < 0$ ,  $V_0 = \mathbb{C}\mathbf{1}$ ,  $V_1 = 0$ ;
- (2)  $(\text{Ker } \mathcal{D}) \cap V_2 = 0$ , where  $\mathcal{D}$  is the linear operator on  $V$  defined by  $\mathcal{D}v = v_{-2}\mathbf{1}$ ;
- (3)  $V = \text{span}\{a_{-m_1}^{(1)} \cdots a_{-m_r}^{(r)} \mathbf{1} \mid r \geq 0, a^{(i)} \in V_2, m_i \geq 1\}$ .

Then there exist a bilinear operation  $*$  on  $V_2$ , uniquely determined by

$$b_0a = \mathcal{D}(a * b) \quad \text{for } a, b \in V_2,$$

and a bilinear form  $\langle \cdot, \cdot \rangle$  on  $V_2$ , uniquely determined by

$$a_3b = \frac{1}{2}\langle a, b \rangle \mathbf{1} \quad \text{for } a, b \in V_2.$$

Furthermore,  $(V_2, *)$  is a Novikov algebra and  $\langle \cdot, \cdot \rangle$  is a symmetric form satisfying (4.11).

*Proof.* As  $V = \oplus_{n \in \mathbb{Z}} V_n$  is a  $\mathbb{Z}$ -graded vertex algebra, we have

$$a_0b \in V_3, \quad a_1b \in V_2, \quad a_2b \in V_1 = 0, \quad a_3b \in V_0 = \mathbb{C}\mathbf{1}, \quad a_nb = 0$$

for  $a, b \in V_2$  and  $n \geq 4$ . From the span property, we have

$$V_3 = \{a_{-2}\mathbf{1} \mid a \in V_2\} = \mathcal{D}V_2.$$

Since  $(\text{Ker } \mathcal{D}) \cap V_2 = 0$ , it follows that  $*$  is a well-defined operation on  $V_2$ . Moreover, using the skew-symmetry of  $V$ , we get

$$b_0a = -a_0b + \mathcal{D}(b_1a), \quad a_3b = b_3a \quad \text{for } a, b \in V_2.$$

It then follows that

$$a * b + b * a = b_1a = a_1b, \quad \langle a, b \rangle = \langle b, a \rangle \quad \text{for } a, b \in V_2. \quad (4.23)$$

Set  $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$  and  $\tilde{\mathcal{D}} = \mathcal{D} \otimes 1 + 1 \otimes \frac{d}{dt}$ . It was known that  $\tilde{V}/\tilde{\mathcal{D}}\tilde{V}$  is a Lie algebra where for  $u, v \in V$ ,  $m, n \in \mathbb{Z}$ ,

$$[\overline{u \otimes t^m}, \overline{v \otimes t^n}] = \sum_{j \geq 0} \binom{m}{j} \overline{u_j v \otimes t^{m+n-j}}.$$

For  $a, b \in V_2$ ,  $m, n \in \mathbb{Z}$ , we have

$$\begin{aligned}
& [a_m, b_n] \\
&= \sum_{i \geq 0} \binom{m}{i} (a_i b)_{m+n-i} \\
&= (a_0 b)_{m+n} + m(a_1 b)_{m+n-1} + \frac{1}{6}m(m-1)(m-2)(a_3 b)_{m+n-3} \\
&= (\mathcal{D}(b * a))_{m+n} + m(a * b + b * a)_{m+n-1} + \frac{1}{6}m(m-1)(m-2)(a_3 b)_{m+n-3} \\
&= -(m+n)(b * a)_{m+n-1} + m(a * b + b * a)_{m+n-1} + \frac{1}{6}m(m-1)(m-2)(a_3 b)_{m+n-3} \\
&= m(a * b)_{m+n-1} - n(b * a)_{m+n-1} + \frac{1}{12}m(m-1)(m-2)\langle a, b \rangle \mathbf{1}_{m+n-3} \\
&= m(a * b)_{m+n-1} - n(b * a)_{m+n-1} + \frac{1}{12}m(m-1)(m-2)\langle a, b \rangle \delta_{m+n,2}.
\end{aligned}$$

From this and the assumption (2), we see that the non-associative algebra  $\tilde{L}(V_2, *)$  defined in Proposition 4.8 is a subalgebra of  $\tilde{V}/\tilde{\mathcal{D}}\tilde{V}$ . In view of Proposition 4.8,  $(V_2, *)$  is a Novikov algebra and  $\langle \cdot, \cdot \rangle$  is a symmetric bilinear form satisfying (4.11).  $\square$

**Remark 4.24.** Let  $V$  be a vertex algebra. Suppose  $v \in \text{Ker } \mathcal{D} \subset V$ . Then

$$\frac{d}{dx}Y(v, x) = Y(\mathcal{D}v, x) = 0,$$

which implies that  $v_n = 0$  for all  $n \neq -1$ . Consequently,  $v$  lies in the center of  $V$  and  $[\mathcal{D}, v_{-1}] = v_{-2} = 0$ . It then follows that  $v_{-1}V$  is an ideal of  $V$ . If  $V$  is  $\mathbb{N}$ -graded and if  $v \in (\text{Ker } \mathcal{D}) \cap V_n$  with  $n \geq 1$ , then  $v_{-1}V$  is a proper ideal. Thus, if  $V$  is a graded simple  $\mathbb{N}$ -graded vertex algebra, we have  $(\text{Ker } \mathcal{D}) \cap V_n = 0$  for all  $n \geq 1$ .

**Remark 4.25.** The bilinear operation  $*$  on  $V_2$  was used by Dijkgraaf in [9] in his study on the genus one partition function, which is controlled by a contact term pre-Lie algebra given in terms of the operator product expansion.

We next discuss quasi vertex operator algebras, or namely Möbius vertex algebras. Fix a basis  $\{L(1), L(0), L(-1)\}$  for  $sl(2, \mathbb{C})$  such that

$$[L(0), L(\pm 1)] = \mp L(\pm 1), \quad [L(1), L(-1)] = 2L(0).$$

A *Möbius vertex algebra* (see [14]) is a  $\mathbb{Z}$ -graded vertex algebra  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ , equipped with a representation of  $sl(2, \mathbb{C})$  on  $V$  such that  $V_n = 0$  for  $n$  sufficiently negative,

$$L(0)|_{V_n} = n \quad \text{for } n \in \mathbb{Z},$$

and

$$[L(-1), Y(v, x)] = Y(L(-1)v, x) = \frac{d}{dx}Y(v, x), \quad (4.26)$$

$$[L(1), Y(v, x)] = Y(L(1)v, x) + 2xY(L(0)v, x) + x^2Y(L(-1)v, x) \quad (4.27)$$

for  $v \in V$ . Note that for a Möbius vertex algebra  $V$ , we have  $L(-1) = \mathcal{D}$  on  $V$ .

We have (cf. [13]):

**Proposition 4.28.** *Let  $V = \oplus_{n \in \mathbb{Z}} V_n$  be a Möbius vertex algebra satisfying all the conditions given in Proposition 4.22. Then  $(V_2, *)$  is a commutative and associative algebra.*

*Proof.* For  $v \in V_2$ , as  $L(1)v \in V_1 = 0$ , we have

$$L(1)L(-1)v = L(-1)L(1)v + 2L(0)v = 4v. \quad (4.29)$$

This implies

$$(\ker L(-1)) \cap V_2 = 0.$$

On the other hand, for  $v \in V_2$ , from [14] we have

$$[L(1), v_m] = (-m + 2)v_{m+1} \quad \text{for } m \in \mathbb{Z}. \quad (4.30)$$

Let  $u, v \in V_2$ . Using (4.29), the definition of  $u * v$ , and (4.30), we get

$$4u * v = L(1)L(-1)(u * v) = L(1)(v_0u) = v_0L(1)u - 2v_1u = 2v_1u,$$

which gives  $u * v = \frac{1}{2}v_1u$ . On the other hand, using skew symmetry we get

$$u_1v = v_1u - L(-1)v_2u + \frac{1}{2}L(-1)^2v_3u + \cdots = v_1u.$$

Therefore  $u * v = v * u$ . This proves that  $(V_2, *)$  is commutative. Consequently,  $(V_2, *)$  is commutative and associative.  $\square$

Furthermore, we have:

**Proposition 4.31.** *Let  $\mathcal{A}$  be a Novikov algebra equipped with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  satisfying (4.11). Then for any  $\ell \in \mathbb{C}$ , the  $\mathbb{Z}$ -graded vertex algebra  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  has a compatible Möbius vertex algebra structure if and only if  $\mathcal{A}$  is commutative and associative.*

*Proof.* The “only if” part follows from Proposition 4.28. For the “if” part we assume that  $\mathcal{A}$  is a commutative and associative algebra with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  satisfying (4.11). First, by Lemma C,  $sl_2$  acts on  $\tilde{L}(\mathcal{A})$  by derivations, where  $sl_2 \cdot \mathbf{c} = 0$  and

$$L(-1 + j)(a \otimes t^n) = (j - n)(a \otimes t^{n+j-1})$$

for  $j = 0, 1, 2$  and for  $a \in \mathcal{A}$ ,  $n \in \mathbb{Z}$ . Then  $sl_2$  acts on the universal enveloping algebra  $U(\tilde{L}(\mathcal{A}))$  as a Lie algebra of derivations. We see that the action of  $sl_2$  preserves the subalgebra  $\tilde{L}(\mathcal{A})_+$ . It follows from the construction of  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  that  $sl_2$  acts on  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  with  $sl_2 \cdot \mathbf{1} = 0$ . For  $a \in \mathcal{A}$ , we have

$$\begin{aligned} [L(-1), Y(a, x)] &= \frac{d}{dx} Y(a, x), \\ [L(0), Y(a, x)] &= x \frac{d}{dx} Y(a, x) + 2Y(a, x), \\ [L(1), Y(a, x)] &= 4xY(a, x) + x^2 \frac{d}{dx} Y(a, x). \end{aligned}$$

Then by Lemma B (in Appendix)  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  is a Möbius vertex algebra.  $\square$

**Remark 4.32.** Let  $\mathcal{A} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$  with a multiplicative operation  $\circ$  given by

$$e_1 \circ e_1 = e_1 + e_2, \quad e_2 \circ e_1 = e_2, \quad e_1 \circ e_2 = e_2 \circ e_2 = 0.$$

This is a noncommutative and nonassociative Novikov algebra. Furthermore, the bilinear form  $\langle \cdot, \cdot \rangle$ , defined by

$$\langle e_1, e_1 \rangle = \frac{1}{12}, \quad \langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = \langle e_2, e_2 \rangle = 0,$$

is (degenerate) symmetric and satisfies (4.11). In view of Proposition 4.31,  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  is not a Möbius vertex algebra. The corresponding Lie algebra  $L(\mathcal{A})$  has been extensively studied in [35].

Next we study  $\phi_\epsilon$ -coordinated modules for vertex algebra  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ . First, we construct certain infinite-dimensional Lie algebras, generalizing Lie algebra  $\tilde{L}(\mathcal{A})$ .

**Lemma 4.33.** *Let  $\mathcal{A}$  be a Novikov algebra and let  $\mathcal{K}$  be a commutative and associative algebra with a derivation  $\partial$ . Define a bilinear operation  $[\cdot, \cdot]$  on  $\mathcal{A} \otimes \mathcal{K}$  by*

$$[a \otimes f, b \otimes g] = ab \otimes (\partial f)g - ba \otimes (\partial g)f \quad (4.34)$$

*for  $a, b \in \mathcal{A}$ ,  $f, g \in \mathcal{K}$ . Then  $(\mathcal{A} \otimes \mathcal{K}, [\cdot, \cdot])$  is a Lie algebra.*

*Proof.* It is straightforward. Alternatively, it follows from a general result in algebraic operad theory (see [29] and [41]) as follows: First, define a new operation  $*$  on  $\mathcal{K}$  by

$$f * g = (\partial f)g \quad \text{for } f, g \in \mathcal{K}.$$

Then  $(\mathcal{K}, *)$  is a right Novikov algebra. Note that from [12] (Theorem 1.3), left Novikov algebras and right Novikov algebras are algebras over binary quadratic operads dual to each other. It follows from [18] (Theorem 2.2.6 (b)) that  $(\mathcal{A} \otimes \mathcal{K}, [\cdot, \cdot])$  is a Lie algebra.  $\square$

In view of Lemma 4.33, for any Novikov algebra  $\mathcal{A}$  and for any integer  $\epsilon$  we have a Lie algebra  $L^\epsilon(\mathcal{A})$ , where

$$L^\epsilon(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] \quad (4.35)$$

as a vector space and the bilinear operation  $[\cdot, \cdot]$  is given by

$$[a \otimes f, b \otimes g] = ab \otimes \left( t^\epsilon \frac{d}{dt} f \right) g - ba \otimes \left( t^\epsilon \frac{d}{dt} g \right) f \quad (4.36)$$

for  $a, b \in \mathcal{A}$ ,  $f, g \in \mathbb{C}[t, t^{-1}]$ .

**Remark 4.37.** Assume that  $\mathcal{A}$  is a commutative Novikov algebra. It can be readily seen that the linear map  $\theta : L(\mathcal{A}) \rightarrow L^\epsilon(\mathcal{A})$  defined by  $\theta(L(a, m)) = L^\epsilon(a, m)$  for  $a \in \mathcal{A}, m \in \mathbb{Z}$  is an isomorphism of Lie algebras. Furthermore, if  $\mathcal{A}$  is unital, that is,  $\mathcal{A}$  is a unital commutative and associative algebra, one can show that  $\tilde{L}(\mathcal{A}) \simeq \tilde{L}^\epsilon(\mathcal{A})$ .

**Remark 4.38.** Let  $\mathcal{A} = \mathbb{C}[z, z^{-1}]$  with a derivation  $z \frac{d}{dz}$ . Define

$$z^i \circ z^j = z^i \left( z \frac{d}{dz} \right) (z^j) = j z^{i+j} \quad \text{for } i, j \in \mathbb{Z}.$$

Then we have a Novikov algebra  $(\mathcal{A}, \circ)$ . Furthermore, for any  $\epsilon \in \mathbb{Z}$  we have a Lie algebra  $L^\epsilon(\mathcal{A})$ . Denote  $L^\epsilon(i, m) = L^\epsilon(z^i, m) \in L^\epsilon(\mathcal{A})$  for  $i, m \in \mathbb{Z}$ . Then

$$[L^\epsilon(i, m), L^\epsilon(j, n)] = (j(m+1-\epsilon) - i(n+1-\epsilon)) L^\epsilon(i+j, m+n) \quad \text{for } i, j, m, n \in \mathbb{Z}.$$

Note that  $L^0(\mathcal{A})$  is isomorphic to the poisson Poisson Lie algebra defined as in (4.14). On the other hand,  $L^1(\mathcal{A})$  is isomorphic to the Lie algebra of area-preserving diffeomorphisms of the two-torus investigated by V. Arnold in [1], which is generated by  $L_m^i$  with  $m, i \in \mathbb{Z}$ , subject to relations

$$[L_m^i, L_n^j] = (jm - in) L_{m+n}^{i+j} \quad \text{for } i, j, m, n \in \mathbb{Z}. \quad (4.39)$$

It was also called the Virasoro-like algebra in [19]. Note that from [10], Lie algebra  $L^1(\mathcal{A})$  is *not* isomorphic to  $L(\mathcal{A})$ .

**Proposition 4.40.** Let  $\mathcal{A}$  be a Novikov algebra with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  satisfying (4.11). Set

$$\tilde{L}^\epsilon(\mathcal{A}) = L^\epsilon(\mathcal{A}) \oplus \mathbb{C} \mathbf{c}_\epsilon, \quad (4.41)$$

where  $\mathbf{c}_\epsilon$  is a nonzero element. For  $a \in \mathcal{A}$ ,  $m \in \mathbb{Z}$ , denote  $L^\epsilon(a, m) = a \otimes t^{m+1-\epsilon}$ . Then  $\tilde{L}^\epsilon(\mathcal{A})$  is a Lie algebra with

$$\begin{aligned} & [L^\epsilon(a, m), L^\epsilon(b, n)] \\ &= (m+1-\epsilon) L^\epsilon(ab, m+n) - (n+1-\epsilon) L^\epsilon(ba, m+n) \\ & \quad + \frac{1}{12} (m+1-\epsilon) m (m-1+\epsilon) \langle a, b \rangle \delta_{m+n, 0} \mathbf{c}_\epsilon \end{aligned} \quad (4.42)$$

for  $a, b \in \mathcal{A}$ ,  $m, n \in \mathbb{Z}$ , and with  $\mathbf{c}_\epsilon$  central.

*Proof.* Define a bilinear form  $(\cdot, \cdot)$  on the Lie algebra  $L^\epsilon(\mathcal{A})$  by

$$(L^\epsilon(a, m), L^\epsilon(b, n)) = \frac{1}{12}(m+1-\epsilon)m(m-1+\epsilon)\langle a, b \rangle \delta_{m+n,0}$$

for  $a, b \in \mathcal{A}$ ,  $m, n \in \mathbb{Z}$ . Notice that

$$(m+1-\epsilon)m(m-1+\epsilon) = m(m^2 - (1-\epsilon)^2),$$

which is an odd function of  $m$ . As  $\langle \cdot, \cdot \rangle$  is symmetric,  $(\cdot, \cdot)$  is skew symmetric.

For cocycle condition, let  $a, b, c \in \mathcal{A}$ ,  $m, n, k \in \mathbb{Z}$ . We have

$$\begin{aligned} & ([L^\epsilon(a, m), L^\epsilon(b, n)], L^\epsilon(c, k)) \\ &= \frac{1}{12}(m+1-\epsilon)(m+n) \left( (m+n)^2 - (1-\epsilon)^2 \right) \langle ab, c \rangle \delta_{m+n+k,0} \\ & \quad - \frac{1}{12}(n+1-\epsilon)(m+n) \left( (m+n)^2 - (1-\epsilon)^2 \right) \langle ba, c \rangle \delta_{m+n+k,0} \\ &= \frac{1}{12}(m-n)(m+n) \left( (m+n)^2 - (1-\epsilon)^2 \right) \langle ab, c \rangle \delta_{m+n+k,0} \\ &= \frac{1}{12}(m^2 - n^2) (k^2 - (1-\epsilon)^2) \langle ab, c \rangle \delta_{m+n+k,0}, \end{aligned}$$

where we used the property  $\langle ab, c \rangle = \langle ba, c \rangle$ . Furthermore, we have

$$\langle bc, a \rangle = \langle a, bc \rangle = \langle ab, c \rangle, \quad \langle ca, b \rangle = \langle c, ab \rangle = \langle ab, c \rangle,$$

and

$$(m^2 - n^2) (k^2 - (1-\epsilon)^2) + (n^2 - k^2) (m^2 - (1-\epsilon)^2) + (k^2 - m^2) (n^2 - (1-\epsilon)^2) = 0.$$

Then the cocycle condition follows immediately. Therefore,  $\tilde{L}^\epsilon(\mathcal{A})$  is a Lie algebra.  $\square$

Note that

$$\tilde{L}^0(\mathcal{A}) = \tilde{L}(\mathcal{A}).$$

For  $a \in \mathcal{A}$ , set

$$L^\epsilon(a, x) = \sum_{n \in \mathbb{Z}} L^\epsilon(a, n) x^{-n-2+2\epsilon} \in \tilde{L}^\epsilon(\mathcal{A})[[x, x^{-1}]]. \quad (4.43)$$

In terms of generating functions the relation (4.42) can be written as

$$\begin{aligned} & [L^\epsilon(a, x_1), L^\epsilon(b, x_2)] \\ &= \left( x_2^\epsilon \frac{\partial}{\partial x_2} L^\epsilon(ba, x_2) \right) x_1^{-1+\epsilon} \delta \left( \frac{x_2}{x_1} \right) \\ & \quad + (L^\epsilon(ab, x_2) + L^\epsilon(ba, x_2)) \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) x_1^{-1+\epsilon} \delta \left( \frac{x_2}{x_1} \right) \\ & \quad + \frac{1}{12} \langle a, b \rangle \mathbf{c}_\epsilon \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^3 x_1^{-1+\epsilon} \delta \left( \frac{x_2}{x_1} \right) \end{aligned} \quad (4.44)$$

for  $a, b \in \mathcal{A}$ .

**Definition 4.45.** An  $\tilde{L}^\epsilon(\mathcal{A})$ -module on which  $\mathbf{c}_\epsilon$  acts as a scalar  $\ell \in \mathbb{C}$  is said to be of level  $\ell$ . An  $\tilde{L}^\epsilon(\mathcal{A})$ -module  $W$  is said to be *restricted* if  $L^\epsilon(a, x)w \in W((x))$  for every  $a \in \mathcal{A}$ ,  $w \in W$ . Denote by  $L_W^\epsilon(a, x)$  the corresponding element of  $\mathcal{E}(W)$ .

As the main result of this section we have:

**Theorem 4.46.** Let  $\ell \in \mathbb{C}$  and let  $W$  be a restricted  $\tilde{L}^\epsilon(\mathcal{A})$ -module of level  $\ell$ . Then there exists a  $\phi_\epsilon$ -coordinated  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ -module structure on  $W$ , which is uniquely determined by  $Y_W(a, x) = L_W^\epsilon(a, x)$  for  $a \in \mathcal{A}$ . On the other hand, let  $(W, Y_W)$  be a  $\phi_\epsilon$ -coordinated  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ -module. Then  $W$  is a restricted  $\tilde{L}^\epsilon(\mathcal{A})$ -module of level  $\ell$ , which is given by  $L_W^\epsilon(a, x) = Y_W(a, x)$  for  $a \in \mathcal{A}$ .

*Proof.* Assume that  $W$  is a restricted  $\tilde{L}^\epsilon(\mathcal{A})$ -module of level  $\ell$ . Set

$$U_W = \text{span}\{L_W^\epsilon(a, x) \mid a \in \mathcal{A}\} \subset \mathcal{E}(W).$$

For  $a, b \in \mathcal{A}$ , from (4.16) we have

$$(x_1 - x_2)^4 [L_W^\epsilon(a, x_1), L_W^\epsilon(b, x_2)] = 0.$$

Thus  $U_W$  is local. By Theorem 3.12,  $U_W$  generates a vertex algebra  $\langle U_W \rangle_{\phi_\epsilon}$  and  $W$  is a faithful  $\phi_\epsilon$ -coordinated  $\langle U_W \rangle_{\phi_\epsilon}$ -module with

$$Y_W(\alpha(x), z) = \alpha(z) \quad \text{for } \alpha(x) \in \langle U_W \rangle_{\phi_\epsilon}.$$

Using the commutation relation of  $\tilde{L}^\epsilon(\mathcal{A})$  we have

$$\begin{aligned} & [Y_W(L_W^\epsilon(a, x), x_1), Y_W(L_W^\epsilon(b, x), x_2)] \\ &= [L_W^\epsilon(a, x_1), L_W^\epsilon(b, x_2)] \\ &= \left( x_2^\epsilon \frac{\partial}{\partial x_2} L_W^\epsilon(ba, x_2) \right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &\quad + (L_W^\epsilon(ab + ba, x_2)) \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &\quad + \frac{1}{12} \langle a, b \rangle \ell \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^3 x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &= \left( x_2^\epsilon \frac{\partial}{\partial x_2} Y_W(L_W^\epsilon(ba, x), x_2) \right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &\quad + Y_W(L_W^\epsilon(ab + ba, x), x_2) \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &\quad + \frac{1}{12} \langle a, b \rangle \ell \left( x_2^\epsilon \frac{\partial}{\partial x_2} \right)^3 x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right). \end{aligned} \tag{4.47}$$

In view of Lemma 3.29, we have

$$\begin{aligned} L_W^\epsilon(a, x)_0^\epsilon L_W^\epsilon(b, x) &= \mathcal{D}L_W^\epsilon(ba, x), & L_W^\epsilon(a, x)_1^\epsilon L_W^\epsilon(b, x) &= L_W^\epsilon(ab + ba, x), \\ L_W^\epsilon(a, x)_3^\epsilon L_W^\epsilon(b, x) &= \frac{1}{2}\ell\langle a, b \rangle 1_W, & L_W^\epsilon(a, x)_j^\epsilon L_W^\epsilon(b, x) &= 0 \end{aligned}$$

for  $j = 2$  and for  $j \geq 4$ . Then by Theorem 3.19 we have

$$\begin{aligned} & [Y_{\mathcal{E}}^\epsilon(L_W^\epsilon(a, x), x_1), Y_{\mathcal{E}}^\epsilon(L_W^\epsilon(b, x), x_2)] \\ &= Y_{\mathcal{E}}^\epsilon(\mathcal{D}L_W^\epsilon(ba, x), x_2) x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) + Y_{\mathcal{E}}^\epsilon(L_W^\epsilon(ab + ba, x), x_2) \left(\frac{\partial}{\partial x_2}\right) x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) \\ & \quad + \frac{1}{12}\langle a, b \rangle \ell \left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) \end{aligned}$$

for  $a, b \in \mathcal{A}$ . This shows that  $\langle U_W \rangle_{\phi_\epsilon}$  is an  $\tilde{L}(\mathcal{A})$ -module of level  $\ell$  with  $L(a, x_1)$  acting as  $Y_{\mathcal{E}}^\epsilon(L_W^\epsilon(a, x), x_1)$  for  $a \in \mathcal{A}$  and

$$L(a, n)1_W = L_W^\epsilon(a, x)_n^\epsilon 1_W = 0 \quad \text{for } a \in \mathcal{A}, n \in \mathbb{N}.$$

From the construction of  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ , there exists an  $\tilde{L}(\mathcal{A})$ -module homomorphism  $\rho$  from  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$  to  $\langle U_W \rangle_{\phi_\epsilon}$  with  $\rho(\mathbf{1}) = 1_W$ . That is,

$$\rho(Y(a, x_1)v) = Y_{\mathcal{E}}^\epsilon(L_W^\epsilon(a, x), x_1)\rho(v) \quad \text{for } a \in \mathcal{A}, v \in V_{\tilde{L}(\mathcal{A})}(\ell, 0).$$

Since the vertex algebra  $\langle U_W \rangle_{\phi_\epsilon}$  is generated by  $L_W^\epsilon(a, x)$  for  $a \in \mathcal{A}$ , it follows that  $\rho$  is a homomorphism of vertex algebras. As  $W$  is a  $\phi_\epsilon$ -coordinated module for  $\langle U_W \rangle_{\phi_\epsilon}$ ,  $W$  is a  $\phi_\epsilon$ -coordinated  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ -module through homomorphism  $\rho$ . Therefore  $W$  is a  $\phi_\epsilon$ -coordinated  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ -module.

On the other hand, let  $(W, Y_W)$  be a  $\phi_\epsilon$ -coordinated  $V_{\tilde{L}(\mathcal{A})}(\ell, 0)$ -module. Using the relations (4.18) and the commutator formula (3.21) for  $\phi_\epsilon$ -coordinated modules for vertex algebras in Theorem 3.19, we have

$$\begin{aligned} & [Y_W(a, x_1), Y_W(b, x_2)] \\ &= Y_W(\mathcal{D}(ba), x_2) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) + Y_W(ab + ba, x_2) \left(x^\epsilon \frac{\partial}{\partial x_2}\right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ & \quad + \frac{\ell}{12}\langle a, b \rangle \left(x_2^\epsilon \frac{\partial}{\partial x_2}\right)^3 x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ &= \left(x^\epsilon \frac{\partial}{\partial x}\right) Y_W(ba, x_2) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) + Y_W(ab + ba, x_2) \left(x^\epsilon \frac{\partial}{\partial x_2}\right) x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \\ & \quad + \frac{\ell}{12}\langle a, b \rangle \left(x_2^\epsilon \frac{\partial}{\partial x_2}\right)^3 x_1^{-1+\epsilon} \delta\left(\frac{x_2}{x_1}\right) \end{aligned}$$

for  $a, b \in \mathcal{A}$ , where we use the fact (see [25])

$$Y_W(\mathcal{D}v, x) = \left(x^\epsilon \frac{\partial}{\partial x}\right) Y_W(v, x) \quad \text{for } v \in V_{\tilde{L}(\mathcal{A})}(\ell, 0).$$



This proves that  $W$  is a restricted module for  $\widetilde{L}^\epsilon(\mathcal{A})$  of level  $\ell$ .  $\square$

## APPENDIX

We here establish some basic results we needed in the main body of the paper.

**Lemma (A).** *Let  $V$  be a vertex algebra equipped with a  $\mathbb{Z}$ -grading  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ . Suppose that  $U$  is a graded subspace such that  $V$  as a vertex algebra is generated by  $U$  and*

$$u_r V_n \subset V_{m+n-r-1} \quad \text{for } u \in U \cap V_m, \quad m, n, r \in \mathbb{Z}. \quad (4.48)$$

*Then  $V$  is a  $\mathbb{Z}$ -graded vertex algebra.*

*Proof.* From the definition we need to prove that for every  $v \in V_m$  with  $m \in \mathbb{Z}$  and for every  $n \in \mathbb{Z}$ ,  $v_n$  is a homogeneous operator of degree  $m - n - 1$ . Let  $K$  be the linear span of homogeneous vectors  $v \in V$  such that

$$\deg v_n = \deg v - n - 1 \quad \text{for all } n \in \mathbb{Z}.$$

Now we must prove  $K = V$ . By assumption we have  $U \subset K$  and it is clear that  $\mathbf{1} \in K$ . Recall the iterate formula: For  $a, b \in V$ ,  $m, n \in \mathbb{Z}$ ,

$$(a_m b)_n = \sum_{i \geq 0} \binom{m}{i} (-1)^i (a_{m-i} b_{n+i} - (-1)^m b_{m+n-i} a_i). \quad (4.49)$$

It follows from this formula that  $K$  is a graded vertex subalgebra. As  $U$  generates  $V$ , we must have  $K = V$ .  $\square$

**Lemma (B).** *Let  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  be a  $\mathbb{Z}$ -graded vertex algebra and an  $sl_2$ -module such that  $sl_2 \cdot \mathbf{1} = 0$ . Suppose that  $U$  is a graded subspace such that  $V$  as a vertex algebra is generated by  $U$  and*

$$\begin{aligned} [L(-1), u_n] &= -n u_{n-1}, & [L(0), u_n] &= (\deg u - n - 1) u_n, \\ [L(1), Y(u, x)] &= Y((L(1) + 2xL(0) + x^2L(-1))u, x) \end{aligned}$$

*for homogeneous  $u \in U$  and for every integer  $n$ . Then  $V$  is a Möbius vertex algebra.*

*Proof.* Just as in the proof of Lemma A, using (4.49) we get  $\deg v_n = \deg v - n - 1$  for every homogeneous vector  $v \in V$  and for every integer  $n$ . It follows that  $L(0)|_{V_m} = m$  for  $m \in \mathbb{Z}$  as

$$L(0)v = L(0)v_{-1}\mathbf{1} = v_{-1}L(0)\mathbf{1} + mv_{-1}\mathbf{1} = mv \quad \text{for } v \in V_m.$$

Note that by assumption, we have  $[L(-1), Y(u, x)] = \frac{d}{dx}Y(u, x)$  for  $u \in U$ . Assume

$$[L(-1), Y(a, x)] = \frac{d}{dx}Y(a, x), \quad [L(-1), Y(b, x)] = \frac{d}{dx}Y(b, x)$$

for some  $a, b \in V$ . Using the vertex-operator form of (4.49), as in [24] (Lemma 3.1.8) we get

$$[L(-1), Y(Y(a, x_0)b, x)] = \frac{\partial}{\partial x} Y(Y(a, x_0)b, x).$$

Then it follows that  $[L(-1), Y(v, x)] = \frac{d}{dx} Y(v, x)$  for all  $v \in V$ .

Next, we consider  $L(1)$ . Set  $A(x) = L(1) + 2xL(0) + x^2L(-1)$ . Suppose

$$[L(1), Y(a, x)] = Y(A(x)a, x), \quad [L(1), Y(b, x)] = Y(A(x)b, x)$$

for some  $a, b \in V$ . Then

$$\begin{aligned} & [L(1), Y(Y(a, x_0)b, x_2)] \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) [L(1), Y(a, x_1)Y(b, x_2)] \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) [L(1), Y(b, x_2)Y(a, x_1)] \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) (Y(A(x_1)a, x_1)Y(b, x_2) + Y(a, x_1)Y(A(x_2)b, x_2)) \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) (Y(A(x_2)b, x_2)Y(a, x_1) + Y(b, x_2)Y(A(x_1)a, x_1)) \\ = & \text{Res}_{x_1} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) [Y(Y(A(x_1)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2)] \\ = & Y(Y(A(x_2 + x_0)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & Y((L(1) + 2x_2L(0) + x_2^2L(-1))Y(a, x_0)b, x_2) \\ = & Y(Y(a, x_0)L(1)b, x_2) + Y(Y((L(1) + 2x_0L(0) + x_0^2L(-1))a, x_0)b, x_2) \\ & + 2x_2Y(Y(a, x_0)L(0)b, x_2) + 2x_2Y(Y((L(0) + x_0L(-1))a, x_0)b, x_2) \\ & + x_2^2Y(Y(a, x_0)L(-1)b, x_2) + x_2^2Y(Y(L(-1)a, x_0)b, x_2) \\ = & Y(Y((L(1) + 2(x_2 + x_0)L(0) + (x_2 + x_0)^2L(-1))a, x_0)b, x_2) \\ & + Y(Y(a, x_0)(L(1) + 2x_2L(0) + x_2^2L(-1))b, x_2) \\ = & Y(Y(A(x_2 + x_0)a, x_0)b, x_2) + Y(Y(a, x_0)A(x_2)b, x_2). \end{aligned}$$

Thus

$$[L(1), Y(Y(a, x_0)b, x_2)] = Y(A(x_2)Y(a, x_0)b, x_2).$$

Then it follows that  $[L(1), Y(v, x)] = Y(A(x)v, x)$  for all  $v \in V$ . Therefore,  $V$  is a Möbius vertex algebra.  $\square$

**Lemma (C).** *Let  $\mathcal{A}$  be a commutative and associative algebra equipped with a symmetric associative bilinear form  $\langle \cdot, \cdot \rangle$  and let  $\tilde{L}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$  be the corresponding Lie algebra with  $\mathbf{c}$  central and with*

$$[a \otimes t^m, b \otimes t^n] = (m - n)(ab \otimes t^{m+n-1}) + \frac{1}{12}m(m-1)(m-2)\langle a, b \rangle \delta_{m+n-2,0}\mathbf{c}$$

for  $a, b \in \mathcal{A}$ ,  $m, n \in \mathbb{Z}$ . Then  $sl_2$  acts on  $\tilde{L}(\mathcal{A})$  as a Lie algebra of derivations with

$$\begin{aligned} L(-1) \cdot (a \otimes t^n) &= -\frac{d}{dt}(a \otimes t^n) = -n(a \otimes t^{n-1}), \\ L(0) \cdot (a \otimes t^n) &= \left(1 - t\frac{d}{dt}\right)(a \otimes t^n) = (1 - n)(a \otimes t^n), \\ L(1) \cdot (a \otimes t^n) &= \left(2t - t^2\frac{d}{dt}\right)(a \otimes t^n) = (2 - n)(a \otimes t^{n+1}) \end{aligned}$$

for  $a \in \mathcal{A}$ ,  $n \in \mathbb{Z}$  and with  $sl_2 \cdot \mathbf{c} = 0$ .

*Proof.* First, we have the required commutation relations

$$\begin{aligned} \left[2t - t^2\frac{d}{dt}, -\frac{d}{dt}\right] &= \left[\frac{d}{dt}, 2t - t^2\frac{d}{dt}\right] = 2 - 2t\frac{d}{dt}, \\ \left[1 - t\frac{d}{dt}, -\frac{d}{dt}\right] &= \left[\frac{d}{dt}, 1 - t\frac{d}{dt}\right] = -\frac{d}{dt}, \\ \left[1 - t\frac{d}{dt}, 2t - t^2\frac{d}{dt}\right] &= -\left[t\frac{d}{dt}, t\left(2 - t\frac{d}{dt}\right)\right] = -t\left(2 - t\frac{d}{dt}\right) = -\left(2t - t^2\frac{d}{dt}\right). \end{aligned}$$

For  $a, b \in \mathcal{A}$ ,  $m, n \in \mathbb{Z}$ , we have

$$\begin{aligned} &[L(-1)(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, L(-1)(b \otimes t^n)] \\ &= -m[a \otimes t^{m-1}, b \otimes t^n] - n[a \otimes t^m, b \otimes t^{n-1}] \\ &= -m(m-1-n)(ab \otimes t^{m+n-2}) - \frac{1}{12}m(m-1)(m-2)(m-3)\delta_{m+n-3,0}\langle a, b \rangle \mathbf{c} \\ &\quad - n(m-n+1)(ab \otimes t^{m+n-2}) - \frac{1}{12}nm(m-1)(m-2)\delta_{m+n-3,0}\langle a, b \rangle \mathbf{c} \\ &= (m-n)(1-m-n)(ab \otimes t^{m+n-2}) \\ &= L(-1)[a \otimes t^m, b \otimes t^n], \end{aligned}$$

$$\begin{aligned}
& [L(1)(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, L(1)(b \otimes t^n)] \\
= & (2-m)[a \otimes t^{m+1}, b \otimes t^n] + (2-n)[a \otimes t^m, b \otimes t^{n+1}] \\
= & (2-m)(m+1-n)(ab \otimes t^{m+n}) + \frac{1}{12}(2-m)(m^3-m)\delta_{m+n-1,0}\langle a, b \rangle \mathbf{c} \\
& + (2-n)(m-n-1)(ab \otimes t^{m+n}) + \frac{1}{12}(2-n)m(m-1)(m-2)\delta_{m+n-1,0}\langle a, b \rangle \mathbf{c} \\
= & (m-n)(3-m-n)(ab \otimes t^{m+n}) \\
= & L(1)[a \otimes t^m, b \otimes t^n].
\end{aligned}$$

This proves that  $L(-1)$  and  $L(1)$  act as derivations. As  $[L(1), L(-1)] = 2L(0)$ ,  $L(0)$  also acts as a derivation.  $\square$

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